

A note on the chromatic polynomials of clique-theta graphs

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July 2, 2012

Abstract

We describe a family of graphs obtained by blowing up all but one of the vertices of a generalised theta graph into cliques. We derive a formula for the chromatic polynomial of an arbitrary member of this family, and use this to show that, if α is a chromatic root of a generalised theta graph, then $p\alpha$ is a chromatic root for all natural numbers p .

We define a generalised theta graph Θ_{m_1, \dots, m_n} to be two vertices joined by n otherwise disjoint paths of length m_1, \dots, m_n . In the study of the location of chromatic roots, generalised theta graphs have been the subject of some interest (see for example [1], [2] and [6]). Most significantly, Sokal [8] used these graphs to prove that chromatic roots are dense in the whole complex plane. In fact, he was able to prove this result using only those theta graphs in which each path between the endpoint vertices is of the same length.

The main aim of this note is to derive a formula for the chromatic polynomials of the graphs obtained by “blowing up” the vertices of generalised theta graphs into cliques of arbitrary size. It is clear that these graphs are chromatically highly diverse, and thus potentially provide a very useful means by which to study general properties of the chromatic polynomial. In particular, the following result — which follows easily from the form of the chromatic polynomials of these graphs — is notable.

Theorem 1. *If α is a non-integer chromatic root of a generalised theta graph, then $p\alpha$ is a chromatic root for all natural numbers p .*

One consequence of this result is a partial proof of the “ $n\alpha$ conjecture” of [3], which proposes that if α is any chromatic root, then $n\alpha$ is also a chromatic

root for any natural number n . It follows immediately from Theorem 1 and Sokal's density result that the set of chromatic roots for which this conjecture holds is dense in the complex plane.

We will need the following basic properties of the chromatic polynomial, the proof of which can be found in any text on the subject.

Proposition 2. *Let G be a simple graph.*

- (i) *Let $G \setminus e$ be the graph obtained from G by deleting the edge e , and let G/e be obtained by contracting the endpoints of e into one vertex and deleting any resulting loops and all parallel edges but one. Then*

$$P_G(x) = P_{G \setminus e}(x) - P_{G/e}(x).$$

If $e \notin E(G)$, then we have:

$$P_G(x) = P_{G+e}(x) + P_{G+e/e}(x),$$

where $G + e$ is the graph is obtained by adding the edge e to G .

- (ii) *If G is a K_n -sum of two subgraphs H_1 and H_2 (that is, if $H_1 \cup H_2 = G$ and $H_1 \cap H_2 = K_n$), then*

$$P_G(x) = \frac{P_{H_1}(x)P_{H_2}(x)}{(x)_n}.$$

We will refer to the two identities in Proposition 2 (i) as, respectively, the *deletion-contraction* and *addition-contraction* identities. Now, we define the *clique-path* $L(a_1, \dots, a_n)$ to be a path of length n , in which the i th vertex has been replaced by a clique of size a_i , and each edge has been replaced by all possible edge between neighbouring cliques. Using Proposition 2 (ii) we can immediately see that its chromatic polynomial is:

$$P_{L(a_1, \dots, a_n)}(x) = \frac{(x)_{a_1+a_2} \cdots (x)_{a_{n-1}+a_n}}{(x)_{a_2} \cdots (x)_{a_{n-1}}}$$

Clique-paths are a subfamily of clique-graphs: graphs consisting of an underlying structure, of which vertices have been blown up into cliques. Another simple clique-graph is defined as follows. The *ring of cliques* $R(a_1, \dots, a_n)$ is a simple n -cycle, the i th vertex of which has been replaced by a clique of size a_i , and each edge of which has been replaced by all possible edges between neighbouring cliques. In [7] Read gives the following general formula for its chromatic polynomial:

$$P_{R(a_1, \dots, a_n)}(x) = (x)_{a_1+a_2} \cdots (x)_{a_n+a_1} \sum_{k=0}^n (-1)^{nk} v_k(x) \left(\prod_{i=1}^n \frac{-(a_i)_k}{(x)_{a_i+k}} \right),$$

where $v_k(x) = \binom{x}{k} - \binom{x}{k-1}$.

Perhaps contrary to first impressions this is indeed a polynomial, as the terms in the denominator of the summation are cancelled by some of the preceding linear factors. Interestingly, a permutation of the $\{a_i\}$ may change the linear factors, but does not affect the final more complicated factor.

Chromatic polynomials of rings of cliques possess some interesting properties. For example, in [4] it is shown that the real part of a non-integer chromatic root of a ring of four cliques is dependent only on the number of vertices in the graph. Rings of cliques were also used in [3] to show that for any quadratic integer α , there is a natural number n such that $\alpha + n$ is a quadratic root.

In the case $a_1 = 1$, the chromatic polynomial of a ring of cliques reduces to the following considerably simpler expression, also discovered — but not published — by Read (a separate derivation is given in [5]):

$$P_{R(1, a_2, \dots, a_n)}(x) = x(x-1)_{a_{n-1}+a_n-1} \left(\prod_{i=2}^{n-2} (x - a_{i+1} - 1)_{a_i-1} \right) r(1, a_2, \dots, a_n), \quad (1)$$

where

$$r(1, a_2, \dots, a_n) = \frac{1}{x} \left(\prod_{i=2}^n (x - a_i) - \prod_{i=2}^n (-a_i) \right).$$

Note that the only (possibly) non-linear factor here is $r(1, a_2, \dots, a_n)$; later we will refer to a factor of this form as the “interesting factor” of a ring of cliques.

We are now in a position of being able to compute the chromatic polynomials of the graphs used in the proof of Theorem 1. These are also clique-graphs, this time having underlying structures of generalised theta graphs; as such they are generalisations of both rings of cliques and theta graphs.

Formally, we define the *clique-theta graph* $T(j, S_1, \dots, S_n, k)$ in the following way: let S_1, \dots, S_n be n non-empty ordered sets of positive integers with $S_i = (a_{i(1)}, a_{i(2)}, \dots, a_{i(m_i)})$ for each $1 \leq i \leq n$. For each set S_i , let $L(1, S_i, k)$ be a clique path with a j -clique at one end, a k -clique at the other end, and clique sizes otherwise determined by the elements of the sets S_i . Then $T(j, S_1, \dots, S_n, k)$ is the graph obtained by identifying the j -cliques at one end of these clique-paths, and the k -cliques at the other.

As might be expected from the complicated formula for general rings of cliques, the chromatic polynomials of these graphs are difficult to compute. However, in a similar way as with rings of cliques, we can considerably simplify this task by specifying that $j = 1$; it is convenient that this condition is actually necessary to guarantee the algebraic properties we need to prove Theorem 1.

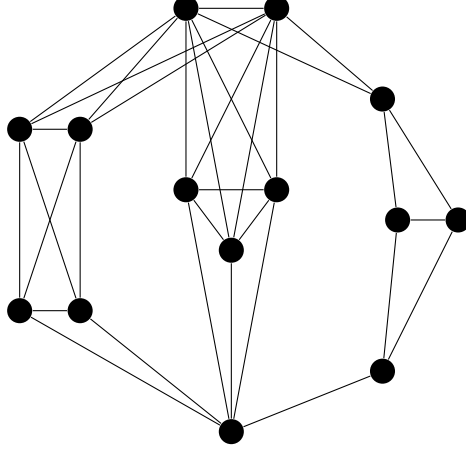


Figure 1: The clique-theta graph $T(1, (2, 2), (3), (1, 2, 1), 2)$

Proposition 3. *The chromatic polynomial of a clique-theta graph $T(1, S_1, \dots, S_n, k)$ is:*

$$\left[(x)_{a_{n(m_n)}+k} \left(\prod_{i=1}^{n-1} (x-k-1)_{a_{i(m_i)}-1} \right) \left(\prod_{i=1}^n \prod_{l=1}^{m_i-1} (x-a_{i(l+1)}-1)_{a_{i(l)}-1} \right) \right] \\ \times \left[\left(k(x-k)^{n-1} \prod_{i=1}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}) \right) + \left(\prod_{i=1}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}, k) \right) \right],$$

where $r(1, a_{i(1)}, \dots, a_{i(m_i)})$ is the interesting factor of a ring of cliques $R(1, a_{i(1)}, \dots, a_{i(m_i)})$.

We will need the following lemma:

Lemma 4. *Let $S_1 = (a_{1(1)}, \dots, a_{1(m_1)})$, and let $\bar{S}_1 = (a_{1(2)}, \dots, a_{1(m_1)})$. Then:*

$$P_{T(1, S_1, \dots, S_n, k)}(x) = \frac{P_{T(1, S_2, \dots, S_n, k)}(x) P_{L(a_{1(1)}, \dots, a_{1(m_1)}, k)}(x)}{(x)_k} \\ - a_{1(1)} \frac{(x)_{a_{1(1)}+a_{1(2)}} P_{T(1, \bar{S}_1, \dots, S_n, k)}(x)}{(x)_{a_{1(2)}+1}},$$

Proof. This follows from a simple application of the deletion-contraction identity. Let v be the singleton endpoint vertex of the clique-theta graph. Deleting the $a_{1(1)}$ edges between v and the $a_{1(1)}$ -clique produces a K_k -sum of $T(1, S_2, \dots, S_n, k)$ and $L(a_{1(1)}, \dots, a_{1(m_1)}, k)$. Contracting one of these edges turns the graph into a $K_{a_{1(2)}+1}$ -sum of $T(1, \bar{S}_1, \dots, S_n, k)$ and $K_{a_{1(1)}+a_{1(2)}}$. As the contraction of any

of these edges produces the same graph, the chromatic polynomial of the latter appears with multiplicity $a_{1(1)}$. \square

We can now proceed by induction on the sizes of the sets S_i .

Proof of Proposition 3. First suppose that the size of each set is 1; that is, suppose $S_i = (a_{i(1)})$ for all i . Let v be the single endpoint vertex, which is in this case connected to all other vertices of the graph apart from those of the k -clique. Note that contracting any added edge between v and the k -clique produces a K_k -sum of $(a_{i(1)} + k)$ -cliques. This graph has chromatic polynomial:

$$f(x) = \frac{\prod_{i=1}^n (x)_{a_{i(1)}+k}}{(x)_k^{n-1}}.$$

Also note that adding all edges between v and the k -clique gives a K_{k+1} -sum of $(a_{i(1)} + k + 1)$ -cliques, having chromatic polynomial:

$$g(x) = \frac{\prod_{i=1}^n (x)_{a_{i(1)}+k+1}}{(x)_{k+1}^{n-1}}.$$

We now apply the contraction-addition identity k times, where “addition” consists of adding an edge between v and the k -clique, and “contraction” consolidates the two vertices between which the new edge is to be added. At every stage the consolidation of these two vertices will produce the graph with chromatic polynomial $f(x)$, and so our final graph will have chromatic polynomial which is a sum of $g(x)$ with k copies of $f(x)$, that is:

$$\begin{aligned} P_{T(1, S_1, \dots, S_n, k)}(x) &= k \frac{\prod_{i=1}^n (x)_{a_{i(1)}+k}}{(x)_k^{n-1}} + \frac{\prod_{i=1}^n (x)_{a_{i(1)}+k+1}}{(x)_{k+1}^{n-1}} \\ &= \left(k(x)_{a_{n(1)}+k} \prod_{i=1}^{n-1} (x-k)_{a_{i(1)}} \right) + \left((x)_{a_{n(1)}+k+1} \prod_{i=1}^{n-1} (x-k-1)_{a_{i(1)}} \right) \\ &= \left((x)_{a_{n(1)}+k} \prod_{i=1}^{n-1} (x-k-1)_{a_{i(1)}-1} \right) \left(k(x-k)^{n-1} + \prod_{i=1}^n (x-a_{i(1)}-k) \right). \end{aligned}$$

Note that $r(1, a_{i(1)}) = 1$ and $r(1, a_{i(1)}, k) = x - a_{i(1)} - k$ for all i . Hence Proposition 3 holds when $|S_i| = 1$ for all i .

These graphs suffice as the base case for the induction, as we can build up any clique-theta graph by starting with one having $|S_i| = 1$ for all i , and systematically increasing the length of the clique-paths. Reordering the S_i does not alter the graph, so for ease of notation we can assume that at each stage the clique-path to which we are adding a new clique is $L(1, S_1, k)$. In a similar way, at each stage we can shift the labelling of the individual cliques up one, so that the new element being added to S_1 is always $a_{1(1)}$.

Thus, by Lemma 4, we need only show that if Proposition 3 holds for $T(1, S_2, \dots, S_n, k)$ and $T(1, \bar{S}_1, \dots, S_n, k)$ then it holds too for $T(1, S_1, \dots, S_n, k)$. So assume that $T(1, S_2, \dots, S_n, k)$ and $T(1, \bar{S}_1, \dots, S_n, k)$ have chromatic polynomials of the stated form, and let:

$$f(x) = (x)_{a_n(m_n)+k} \left(\prod_{i=1}^{n-1} (x - k - 1)_{a_i(m_i)-1} \right) \left(\prod_{i=1}^n \prod_{l=1}^{m_i-1} (x - a_{i(l+1)} - 1)_{a_{i(l)}-1} \right).$$

Then removing $f(x)$ as a factor from

$$\frac{P_{T(1, S_2, \dots, S_n, k)}(x) P_{L(a_1(1), \dots, a_1(m_1), k)}(x)}{(x)_k}$$

and

$$\frac{(x)_{a_1(1)+a_1(2)} P_{T(1, \bar{S}_1, \dots, S_n, k)}(x)}{(x)_{a_1(2)+1}}$$

leaves, respectively:

$$\begin{aligned} & \left(\prod_{l=2}^{m_1} (x - a_{1(l)}) \right) \left(k(x - k)^{n-1} \prod_{i=2}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}) \right) \\ & + (x - k) \left(\prod_{l=2}^{m_1} (x - a_{1(l)}) \right) \left(\prod_{i=2}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}, k) \right), \end{aligned}$$

and:

$$\begin{aligned} & k(x - k)^{n-1} r(1, a_{1(2)}, \dots, a_{1(m_1)}) \left(\prod_{i=2}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}) \right) \\ & + r(1, a_{1(2)}, \dots, a_{1(m_1)}, k) \left(\prod_{i=2}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}, k) \right). \end{aligned}$$

By Lemma 4, the interesting factor of the chromatic polynomial of $T(1, S_1, \dots, S_n, k)$ is obtained by subtracting $a_{1(1)}$ times the latter from the former, giving:

$$\begin{aligned} & \left(k(x - k)^{n-1} \prod_{i=2}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}) \right) \left[\left(\prod_{l=2}^{m_1} (x - a_{1(l)}) \right) - a_{1(1)} r(1, a_{1(2)}, \dots, a_{1(m_1)}) \right] \\ & + \left(\prod_{i=2}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}, k) \right) \left[(x - k) \left(\prod_{l=2}^{m_1} (x - a_{1(l)}) \right) - a_{1(1)} r(1, a_{1(2)}, \dots, a_{1(m_1)}, k) \right] \end{aligned}$$

The proof is completed by noting that:

$$\left(\prod_{l=2}^{m_1} (x - a_{1(l)}) \right) - a_{1(1)} r(1, a_{1(2)}, \dots, a_{1(m_1)}) = r(1, a_{1(a)}, \dots, a_{1(m_1)}).$$

□

Theorem 1 now follows as a corollary of the following result.

Proposition 5. *If $S_i = (a_{i(1)}, \dots, a_{i(m_i)})$, let $pS_i = (pa_{i(1)}, \dots, pa_{i(m_i)})$, and let α be a non-integer chromatic root of the clique-theta graph $T(1, S_1, \dots, S_n, k)$. Then for any natural number p , $p\alpha$ is a chromatic root of $T(1, pS_1, \dots, pS_n, pk)$.*

Proof. We need only consider the non-linear factor of the chromatic polynomial. For $T(1, S_1, \dots, S_n, k)$ this is, by Proposition 3:

$$\left[k(x - k)^{n-1} \prod_{i=1}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}) \right] + \left[\prod_{i=1}^n r(1, a_{i(1)}, \dots, a_{i(m_i)}, k) \right]. \quad (2)$$

Expanding the interesting factors of the rings of cliques, this becomes:

$$\begin{aligned} & \left[k(x - k)^{n-1} \prod_{i=1}^n \frac{1}{x} \left(\prod_{l=1}^{m_i} (x - a_{i(l)}) - \prod_{l=1}^{m_i} (-a_{i(l)}) \right) \right] \\ & + \left[\prod_{i=1}^n \frac{1}{x} \left((x - k) \prod_{l=1}^{m_i} (x - a_{i(l)}) + k \prod_{l=1}^{m_i} (-a_{i(l)}) \right) \right]. \quad (3) \end{aligned}$$

For $T(1, pS_1, \dots, pS_n, pk)$, we have:

$$\begin{aligned} & \left[pk(x - pk)^{n-1} \prod_{i=1}^n \frac{1}{x} \left(\prod_{l=1}^{m_i} (x - pa_{i(l)}) - \prod_{l=1}^{m_i} (-pa_{i(l)}) \right) \right] \\ & + \left[\prod_{i=1}^n \frac{1}{x} \left((x - pk) \prod_{l=1}^{m_i} (x - pa_{i(l)}) + pk \prod_{l=1}^{m_i} (-pa_{i(l)}) \right) \right]. \quad (4) \end{aligned}$$

Let $s = \sum_{i=1}^n (m_i + 1)$. Then dividing (4) by p^s gives:

$$\begin{aligned} & \left[k(x/p - k)^{n-1} \prod_{i=1}^n \frac{1}{x} \left(\prod_{l=1}^{m_i} (x/p - a_{i(l)}) - \prod_{l=1}^{m_i} (-a_{i(l)}) \right) \right] \\ & + \left[\prod_{i=1}^n \frac{1}{x} \left((x/p - k) \prod_{l=1}^{m_i} (x/p - a_{i(l)}) + k \prod_{l=1}^{m_i} (-a_{i(l)}) \right) \right]. \quad (5) \end{aligned}$$

If α is a zero of (3), then $p\alpha$ is a zero of (5).

□

Proof of Theorem 1. Let Θ_{m_1, \dots, m_n} be a generalised theta graph having a non-integer chromatic roots α . Θ_{m_1, \dots, m_n} is the same graph as $T(1, S_1, \dots, S_n, 1)$, in which for each i , $S_i = (1, 1, \dots, 1)$ and $|S_i| = m_i - 2$. By Proposition 5, $p\alpha$ is a chromatic root of $T(1, pS_1, \dots, pS_n, p)$. □

Acknowledgements

This paper was written while under the supervision of Peter Cameron at Queen Mary, University of London. I would like to thank Prof. Cameron for suggesting this field of research, and for his helpful comments and advice. My thanks also go to the EPSRC for financial support for the duration of my doctoral studies.

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